



A proof of uniqueness of the Taub-bolt instanton

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Abstract

We show that the Riemannian Schwarzschild and the “Taub-bolt” instanton solutions are the only spaces $(\mathcal{M}, g_{\mu\nu})$ such that:

- \mathcal{M} is a four-dimensional, simply connected manifold with a Riemannian, Ricci-flat C^2 -metric $g_{\mu\nu}$ which admits (at least) a 1-parameter group μ_τ of isometries without isolated fixed points on \mathcal{M} .
- The quotient $(\mathcal{M} \setminus \mathcal{L}_{\mathcal{M}})/\mu_\tau$ (where $\mathcal{L}_{\mathcal{M}}$ is the set of fixed points of μ_τ) is an asymptotically flat manifold, and the length of the Killing field corresponding to μ_τ tends to a constant at infinity.

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1. Introduction

Attempts of estimating the path integral of Quantum Gravity via the stationary phase approximation motivated the study of “instantons”, i.e. Riemannian, Ricci-flat metrics which are regular everywhere. Gibbons and Hawking [1] distinguished instantons having at least a one-parameter group of isometries according to whether the isometry has isolated fixed points (“nuts”) or two-dimensional subsets of fixed points (“bolts”). The mathematical analogy between instantons with fixed points and Lorentzian solutions with Killing horizons

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raises the question whether the known uniqueness proofs for stationary black hole solutions can be carried over to the Riemannian case. In fact, for asymptotically flat solutions with up to two nuts and no bolts, a uniqueness result for the Riemannian Kerr metric has been obtained [2] by adapting and generalizing Israel's proof of uniqueness of the Schwarzschild solution [3].

The present work concerns the problem of finding uniqueness results for the 1-parameter family of Schwarzschild instantons and for the 1-parameter family of so-called Taub-bolt instantons (found by Page [4] by "Euclideanizing" the Taub-NUT spacetime [5,27]) which read as follows, respectively,

$$ds^2 = \frac{r-m}{r+m} d\tau^2 + \frac{r+m}{r-m} (dr^2 + (r^2 - m^2) d\Omega^2), \quad (1)$$

$$ds^2 = \frac{(r-2|n|)(r-|n|/2)}{(r^2-n^2)} (d\tau + 2n \cos \theta d\phi)^2 + (r^2-n^2) \left(\frac{dr^2}{(r-2|n|)(r-|n|/2)} + d\Omega^2 \right), \quad (2)$$

where $m \geq 0$ and $n \neq 0$ are constants and $d\Omega^2$ is the round metric on the 2-sphere. In (1) the subcase $m = 0$ is just the Euclidean metric on R^4 whereas for $m > 0$ the coordinate τ is periodic with range $0 \leq \tau < 8\pi m$, and $r \geq m$. Regarding (2), τ is periodic with range $0 \leq \tau < 8\pi n$, and $r \geq 2|n|$. These spaces are probably unique in the following sense.

Conjecture 1. *The Riemannian Schwarzschild and the "Taub-bolt" instanton solutions are the only spaces $(\mathcal{M}, g_{\mu\nu})$ such that:*

- C1. \mathcal{M} is a 4-dimensional manifold with a Riemannian, Ricci-flat C^2 -metric $g_{\mu\nu}$ which admits (at least) a 1-parameter group μ_τ of isometries.
- C2. μ_τ has no isolated fixed points on \mathcal{M} .
- C3. \mathcal{M} is simply connected.
- C4. $(\mathcal{M}, g_{\mu\nu})$ is asymptotically flat (AF) or asymptotically locally flat (ALF), and the trajectories of μ_τ have bounded length at infinity.

In defining AF and ALF we may follow [6,28]. (Note that according to this definition, flat space is not AF but asymptotically Euclidean).

While it would be desirable to prove this conjecture, we have obtained a closely related uniqueness result. To formulate the latter we introduce the set $\mathcal{N} = \pi(\mathcal{M} \setminus \mathcal{L}_{\mathcal{M}})$ of non-trivial orbits of μ_τ , where $\mathcal{L}_{\mathcal{M}}$ denotes the set of fixed points of μ_τ and $\pi: \mathcal{M} \rightarrow \mathcal{M}/\mu_\tau$ is the canonical projection. In general, the space \mathcal{N} of Killing orbits is what has been called a V-manifold [8], a Satake-manifold [9] or an orbifold [10]. However, in some cases, (such as for Schwarzschild and Taub-bolt), \mathcal{N} actually has the structure of a (standard) manifold. In this case there exists a one-to-one correspondence between tensors on \mathcal{N} and tensors on $\mathcal{M} \setminus \mathcal{L}_{\mathcal{M}}$ which are orthogonal with respect to every index to the Killing field ξ^μ corresponding to μ_τ , and have vanishing Lie derivative along ξ^μ . In particular, the symmetric covariant tensor $g_{\mu\nu} - V^{-2} \xi_\mu \xi_\nu$ on \mathcal{M} (where $V = (g_{\mu\nu} \xi^\mu \xi^\nu)^{1/2}$ is the norm

of ξ^μ) can be “pulled back” to give a metric on \mathcal{N} which we call g_{ij} . With this metric (\mathcal{N}, g_{ij}) is a Riemannian manifold.

Our theorem reads as follows.

Theorem 1. *The Riemannian Schwarzschild and the Taub-bolt instanton solutions are the only spaces $(\mathcal{M}, g_{\mu\nu})$ such that:*

- T1. *\mathcal{M} is a 4-dimensional manifold with a Riemannian, Ricci-flat C^2 -metric $g_{\mu\nu}$ which admits (at least) a 1-parameter group μ_τ of isometries. Moreover, the set \mathcal{N} of non-trivial orbits of μ_τ is a manifold.*
- T2. *μ_τ has no isolated fixed points on \mathcal{M} .*
- T3. *\mathcal{M} is simply connected.*
- T4. *(\mathcal{N}, g_{ij}) is asymptotically flat (AF) and the norm of the Killing field corresponding to μ_τ tends to a constant at infinity.*

Note that conditions C2–T2 and C3–T3 agree whereas the other conditions of the theorem have similar but different counterparts in the conjecture. In particular, in T1 we have included the requirement that \mathcal{N} is a manifold. Actually, this latter requirement is probably spurious due to the strong condition T2. We will come back to this point in Section 5. As to T4, the notion of asymptotic flatness for (\mathcal{N}, g_{ij}) will be defined in Section 3. We remark that condition T4 allows $(\mathcal{M}, g_{\mu\nu})$ to be flat Euclidean space, in contrast to C4.

Our requirement on the metric to be C^2 permits the introduction of harmonic coordinates, in terms of which the condition of Ricci flatness is an elliptic equation for the metric, and so the latter is analytic. An elementary result (Section VI, Proposition 1.1. of [11]) shows that in this case the Killing field ξ^μ is C^∞ . But since ξ^μ also satisfies an elliptic equation in Ricci flat spaces, it is analytic as well. It is then also possible to find an analytic atlas on \mathcal{M} of the form (t, x^i) where t is a function of the group parameter τ , and x^i are coordinates on \mathcal{N} [12]. It will turn out to be convenient to use this atlas in particular in the asymptotic analysis (Section 3).

Our proof follows the method of Bunting and Masood-ul-Alam for proving uniqueness of the Schwarzschild solution among the static, asymptotically flat vacuum black holes [13] with bifurcate horizons. They construct a complete space $\overline{\mathcal{N}}$ by gluing together two copies of a $t = \text{const.}$ slice along the bifurcation surface of the horizons, and by performing a 1-point compactification of one of the ends. By a suitable conformal transformation, $\overline{\mathcal{N}}$ can be endowed with a metric of vanishing mass, which is sufficiently regular such that the limiting case of the positive mass theorem [14] can be applied. This yields that $\overline{\mathcal{N}}$ is flat and the original space is Schwarzschild.

In our case the orbit space \mathcal{N} takes the role of the $t = \text{const.}$ slices and the bolts take the role of the bifurcation surfaces. To follow the strategy of [13] we first (Section 2) have to study carefully the local geometry of \mathcal{N} in the neighbourhood of a bolt. We then (Section 3) perform a detailed analysis of the asymptotic properties of \mathcal{N} . Section 4 contains a preliminary result on the global geometry and the proof of the theorem. In Section 5 we discuss the options of proving the conjecture stated above, and of improving or generalizing our theorem by relaxing conditions T1 and T2.

2. Properties of bolts

The set $\mathcal{L}_{\mathcal{M}}$ of fixed points of μ_{τ} has the following structure (cf. Section 2 of [1]). If $q \in \mathcal{L}_{\mathcal{M}}$ is isolated it is called a “nut” (after the Euclidean Taub-NUT metric [15]). We exclude nuts by assumption T2 of Theorem 1. The remaining possibility is that each connected component of $\mathcal{L}_{\mathcal{M}}$ is a two-surface called “bolt”. At every $q \in \mathcal{L}_{\mathcal{M}}$ the differential $\mu_{\tau*}$ acts as the identity on a 2-dimensional subspace $T_q^- \mathcal{M}$ of the tangent space $T_q \mathcal{M}$ and as a rotation on the orthogonal subspace $T_q^+ \mathcal{M}$, with period $\tau_+ = 2\pi/\kappa$. κ (called the “gravity” of the bolt) is constant on each bolt and also appears in the matrix representation of $\nabla_{\mu} \xi_{\nu}$ in an orthonormal frame (cf. Theorem 5.3 of [16]).

In a normal neighbourhood U_q of a point q of a bolt, we choose normal coordinates $\{z^{\alpha}\}$ as follows. Choose a basis of orthonormal vectors $\{\vec{v}_{\alpha}\}$ in $T_q \mathcal{M}$ such that $\{\vec{v}_0, \vec{v}_1\}$ span $T_q^+ \mathcal{M}$ and $\{\vec{v}_2, \vec{v}_3\}$ span $T_q^- \mathcal{M}$. The normal coordinates $\{z^{\alpha}\}$ of a point q are $\exp(z^{\alpha} \vec{v}_{\alpha}) = q$. The commutativity of μ_{τ} and the exponential map, viz. $\exp(\mu_{\tau*} X) = \mu_{\tau}(\exp X)$ for $X \in T_q \mathcal{M}$ implies that the action of μ_{τ} is linear in these coordinates, i.e. $\mu_{\tau}^{\alpha}(z) = (\mu_{\tau*})^{\alpha}_{\beta}(q) z^{\beta}$. For the Killing vector we get

$$\xi^{\alpha}(z) = \left. \frac{d}{d\tau} \mu_{\tau}^{\alpha}(z) \right|_{\tau=0} = \kappa(z^1 \partial_{z^0} - z^0 \partial_{z^1}). \tag{3}$$

We now obtain the form of the metric near a bolt in normal coordinates.

Lemma 1. *Let q be an arbitrary point on a bolt and U_q a normal neighbourhood of q . Consider an arbitrary C^2 -metric $g_{\alpha\beta}$ defined on U_q which is invariant under the action of μ_{τ} . Then, in the normal coordinate system introduced above, the metric takes the form (in matrix notation)*

$$g(z) = \left(\begin{array}{cc} S^T(z) & 0 \\ 0 & I_2 \end{array} \right) \left(\begin{array}{cc} A & B \\ B^T & C \end{array} \right) \Big|_{\rho(z), z^2, z^3} \left(\begin{array}{cc} S(z) & 0 \\ 0 & I_2 \end{array} \right), \tag{4}$$

where I_2 is the 2×2 unit matrix, S is the rotation

$$S(z) = \frac{1}{\rho(z)} \left(\begin{array}{cc} z^0 & z^1 \\ -z^1 & z^0 \end{array} \right), \tag{5}$$

$\rho(z) \equiv +\sqrt{(z^0)^2 + (z^1)^2}$, and the 2×2 matrices A, B, C are C^2 functions of three variables ρ, z^2 and z^3 in a domain $\rho \geq 0$. Furthermore,

$$\begin{aligned} A(\rho, z^2, z^3) &= a(z^2, z^3) I_2 + O(\rho^2), \\ B(\rho, z^2, z^3) &= \rho B^{(1)}(z^2, z^3) + O(\rho^2), \\ C(\rho, z^2, z^3) &= C^{(0)}(z^2, z^3) + O(\rho^2), \end{aligned} \tag{6}$$

where $a(z^2, z^3)$ is a function with $a(0, 0) = 1$, and $C^{(0)}(0, 0) = I_2$.

Proof. The action of the isometry group in the normal coordinates $\{z^{\alpha}\}$ is $\mu_{\tau}(z)^{\alpha} = H_{\tau}^{\alpha}_{\beta} z^{\beta}$, where H_{τ} is (in matrix notation)

$$H_\tau = \begin{pmatrix} R_\tau & 0 \\ 0 & I_2 \end{pmatrix}, \quad R_\tau = \begin{pmatrix} \cos(\kappa\tau) & \sin(\kappa\tau) \\ -\sin(\kappa\tau) & \cos(\kappa\tau) \end{pmatrix}.$$

Invariance of the metric under μ_τ means $g_{\alpha\beta}(z) = (\mu_\tau^\mu(z))_{,\alpha}(\mu_\tau^\nu(z))_{,\beta}g_{\mu\nu}(\mu_\tau(z))$, where comma denotes partial derivative. Hence $g(H_\tau z) = H_\tau g(z) H_\tau^T$, where $(H_\tau z)^\alpha = H_\tau^\alpha_\beta z^\beta$. The matrix $S(z)$ is defined only at points with $\rho(z) \neq 0$ (although the expression (4) in the lemma makes sense also at $\rho(z) = 0$ because of the expansion (6)). Let us define a matrix $\tilde{g}(z)$ by

$$\tilde{g}(z) = \begin{pmatrix} S(z) & 0 \\ 0 & I_2 \end{pmatrix} g(z) \begin{pmatrix} S^T(z) & 0 \\ 0 & I_2 \end{pmatrix} \quad \text{for } z \text{ such that } \rho(z) \neq 0.$$

Using the identity $S(H_\tau z) = S(z)R^{-1}(\tau)$, the invariance equation becomes simply $\tilde{g}(H_\tau z) = \tilde{g}(z)$ at points where $\rho(z) \neq 0$. This equation motivates the definition of 2×2 matrices A , B and C , which are functions of three variables ρ, z^2, z^3 on a domain where $\rho \geq 0$, through the equation

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \Big|_{\rho, z^2, z^3} = g(z)|_{z^0=\rho, z^1=0, z^2, z^3}, \quad \text{where } \rho \geq 0. \tag{7}$$

Since the right-hand side is a C^2 function of its arguments, so are A , B and C . Using the form of S and the invariance equation, we obtain for any $\rho < 0$

$$g(z)|_{z^0=\rho, z^1=0, z^2, z^3} = \begin{pmatrix} A & -B \\ -B^T & C \end{pmatrix} \Big|_{-\rho, z^2, z^3}. \tag{8}$$

This equation together with the continuity of $g(z)$ at $\rho(z) = 0$ implies that $B(0, z^2, z^3) = 0$. Let now $a, b = 0, 1$ and $A, B = 2, 3$. Using again the form of $S(z)$ and the invariance equation, we have, for $\rho > 0$,

$$g_{ab}(z)|_{z^0=0, z^1=\rho, z^2, z^3} = \begin{pmatrix} A_{11} & -A_{01} \\ -A_{01} & A_{00} \end{pmatrix} \Big|_{\rho, z^2, z^3}.$$

The limit $\rho \rightarrow 0$ in this expression makes sense, and using definition (7) we obtain $A(0, z^2, z^3) = a(z^2, z^3)I_2$ where $a(z^2, z^3)$ is a C^2 function. Taking partial derivatives of (7) and (8) with respect to ρ , performing the limit $\rho \rightarrow 0$ and using that $g(z)$ has continuous derivatives, we easily get $\partial_\rho A|_{\rho=0} = 0$ and $\partial_\rho C|_{\rho=0} = 0$. Noticing that $g_{\mu\nu}(z = 0) = \delta_{\mu\nu}$, the lemma follows by Taylor expansion. \square

We denote the set of all orbits of μ_τ (including its fixed points \mathcal{L}_M) by $\widehat{\mathcal{N}}$, and define $\pi(\mathcal{L}_M) = \mathcal{L}_{\widehat{\mathcal{N}}}$. The latter set has the following structure.

Lemma 2. *Let $(\mathcal{M}, g_{\mu\nu})$ satisfy conditions T1 and T2 of Theorem 1.*

Then $\mathcal{L}_{\widehat{\mathcal{N}}}$ is a smooth, two-dimensional boundary of \mathcal{N} . Moreover, g_{ij} has a C^2 -extension to $\mathcal{L}_{\widehat{\mathcal{N}}}$ and $\mathcal{L}_{\widehat{\mathcal{N}}}$ is totally geodesic in $(\widehat{\mathcal{N}}, g_{ij})$.

Proof. In a normal neighborhood of a point q on a bolt we use normal coordinates $\{z^a\}$ as introduced above Lemma 1. Let $a, b = 0, 1$; $A, B = 2, 3$, and define $y^0 = z^2, y^1 = z^3$. Using the form of the metric in Lemma 1 we easily obtain, at points where $\rho(z) \neq 0$,

$$\begin{aligned} \xi^a \xi^b g_{ab} &= \kappa^2 \rho^2 A_{11}, & g_{ab} \xi^a dz^b &= -\kappa \rho A_{1b} \alpha^b, & g_{aB} \xi^a dz^B &= -\kappa \rho B_{1b} dy^b, \\ g_{ab} dz^a dz^b &= A_{ab} \alpha^a \alpha^b, & g_{aB} dz^a dz^B &= B_{ab} \alpha^a dy^b, \end{aligned}$$

where we have introduced two one-forms $\alpha^0 = d\rho = (z^0 dz^0 + z^1 dz^1)/\rho$ and $\alpha^1 = (-z^1 dz^0 + z^0 dz^1)/\rho$. It is now straightforward to show that the symmetric tensor $ds_g^2 = (g_{\mu\nu} - V^{-2} \xi_\mu \xi_\nu) dz^\mu dz^\nu$ takes the form

$$\begin{aligned} ds_g^2 &= \frac{\det A}{A_{11}} d\rho^2 + 2 \left(B_{0a} - \frac{A_{01}}{A_{11}} B_{1a} \right) d\rho dy^a \\ &+ \left(C_{ab} - \frac{B_{1a} B_{1b}}{A_{11}} \right) dy^a dy^b. \end{aligned} \tag{9}$$

Since ρ and y^a are constant along the Killing orbits, they are suitable coordinates on $\widehat{\mathcal{N}}$ (ρ is of course restricted to be non-negative) and therefore the metric g_{ij} on \mathcal{N} is given exactly by expression (9), where now $d\rho$ is no longer α^0 but the differential of a coordinate. The boundary of $\widehat{\mathcal{N}}$ which is given locally by $\rho = 0$ is two-dimensional and coincides with $\mathcal{L}_{\widehat{\mathcal{N}}}$ by construction. Lemma 1 implies $ds_g^2|_{\rho=0} = a(y^c) d\rho^2 + C_{ab}^{(0)}(y^c) dy^a dy^b$. Since A, B and C are C^2 functions of ρ, z^2 and z^3 , it follows that the metric g_{ij} extends at least in a C^2 manner to $\rho = 0$. A trivial calculation shows that the second fundamental form of $\mathcal{L}_{\widehat{\mathcal{N}}}$ in $(\widehat{\mathcal{N}}, g_{ij})$ vanishes, which is equivalent to $\mathcal{L}_{\widehat{\mathcal{N}}}$ being totally geodesic. Finally, the smoothness of $\widehat{\mathcal{N}}$ follows from the smoothness of the geodesics. \square

3. Asymptotic structure

Here we first define asymptotic flatness of the manifold (\mathcal{N}, g_{ij}) (contained in assumption T4 of Theorem 1). We then use assumption T1, in particular Ricci flatness, together with asymptotic properties, to introduce the “twist” (-scalar) of the Killing field on an “end” \mathcal{N}^∞ . Next, we adapt from [17] two technical lemmas (Lemmas 3 and 4 below) on the inversion of a certain elliptic operator and of the flat Laplacian. These results will be used to establish falloff properties of the metric on \mathcal{N} and of quantities constructed from the norm and the twist of ξ^μ (Lemma 5), and to show the existence of a compactification of the end of \mathcal{N} (Lemma 6). Most parts of the proofs of these lemmas parallel closely the treatment of the Lorentzian case [18–20].

Definition 1. The manifold (\mathcal{N}, g_{ij}) is called asymptotically flat iff:

- (1) The “end” $\mathcal{N}^\infty = \widehat{\mathcal{N}} \setminus \{\text{a compact set}\}$ is diffeomorphic to $R^3 \setminus B$, where B is a ball. (The compact set is chosen appropriately, in particular, sufficiently large).
- (2) On \mathcal{N}^∞ there are coordinates such that

$$g_{ij} - \delta_{ij} = O^2(r^{-\delta}) \quad \text{for some } \delta > 0. \tag{10}$$

(A function $f(x^i)$ is said to be $O^k(r^\alpha)$, $k \in \mathbb{N}$, if $f(x^i) = O(r^\alpha)$, $\partial_j f(x^i) = O(r^{\alpha-1})$ a.s.o. for all derivatives up to and including the k th ones.)

This definition might appear over-restrictive as it combines the topological condition 1 with the purely asymptotic condition 2. We adopt this definition not only because it follows common practice [6,28] but also because condition 1 is really required to prove the theorem in Section 4. In particular, we require here that \mathcal{N} has a single end. Similarly, we believe that the definitions of AF and of ALF as given in [6,28] will be required to prove the conjecture stated in the introduction.

On the whole of \mathcal{M} we can introduce the twist vector $\omega_\mu = \epsilon_{\mu\nu\sigma\tau} \xi^\nu \nabla^\sigma \xi^\tau$ of the Killing vector ξ^μ . ($\epsilon_{\mu\nu\sigma\tau}$ is antisymmetric and $\epsilon_{0123} = \sqrt{\det g}$). Since $\xi^\mu \omega_\mu = 0$ and since ξ^μ commutes with ω_μ , ω_μ and hence also $\nabla_{[\mu} \omega_{\nu]}$ can be projected to a vector ω_i and to its curl $\partial_{[i} \omega_{j]}$ on \mathcal{N} [23]. From the condition of Ricci-flatness, i.e. $R_{\mu\nu} = 0$, we find that ω_μ and ω_i are curl free, i.e.

$$\nabla_{[\mu} \omega_{\nu]} = 0, \quad \partial_{[i} \omega_{j]} = 0. \tag{11}$$

For the rest of this section we restrict ourselves to the end \mathcal{N}^∞ . Since the latter is simply connected, there exists a scalar ω there, defined up to an additive constant, such that $\partial_i \omega = \omega_i$. Denoting by \mathcal{D} the covariant derivative and by \mathcal{R}_{ij} the Ricci tensor with respect to g_{ij} , we can decompose the condition $R_{\mu\nu} = 0$ as follows (by changing signs appropriately in the corresponding Lorentzian result [23])

$$\mathcal{R}_{ij} = V^{-1} \mathcal{D}_i \mathcal{D}_j V - \frac{1}{2} V^{-4} \{ \omega_i \omega_j - g_{ij} g^{kl} \omega_k \omega_l \}, \tag{12}$$

$$\mathcal{D}^2 V = \frac{1}{2} V^{-3} g^{ij} \omega_i \omega_j, \tag{13}$$

$$\mathcal{D}^2 \omega = 3V^{-2} g^{ij} \omega_i \mathcal{D}_j V, \tag{14}$$

and it follows from (12) and (13) that the Ricci scalar \mathcal{R} with respect to g_{ij} can be expressed as

$$\mathcal{R} = \frac{3}{2} V^{-4} g^{ij} \omega_i \omega_j. \tag{15}$$

By condition T4, $\mathcal{R} = O(r^{-2-\delta})$, and by rescaling ξ^μ suitably, we can achieve that $V \rightarrow 1$. From (15) we obtain $\omega_i = O(r^{-1-\epsilon})$ and so we can adjust the additive constant in ω such that

$$\omega = O^1(r^{-\epsilon}) \quad \text{for some positive } \epsilon. \tag{16}$$

To analyze the full set of field equations it is convenient to introduce some more notation. We will employ the metric $\gamma_{ij} = V^2 g_{ij}$ on \mathcal{N}^∞ and we denote by D_i and R_{ij} the covariant derivative and the Ricci tensor with respect to γ_{ij} . We also introduce

$$v_\pm = \frac{1 - V^2 \mp \omega}{1 + V^2 \pm \omega}, \tag{17}$$

$$\Theta = 1 - v_+ v_- = \frac{4V^2}{(1 + V^2 + \omega)(1 + V^2 - \omega)}. \tag{18}$$

By the asymptotic properties discussed above, the fields v_{\pm} and Θ exist on \mathcal{N}^{∞} , and $0 < \Theta < 1$. Finally we define the vector field $A_i = \frac{1}{2}(v_+ D_i v_- - v_- D_i v_+)$. On \mathcal{N}^{∞} , the conditions $R_{\mu\nu} = 0$ yield

$$D_i(\Theta^{-1} D^i v_{\pm}) = \pm 2\Theta^{-2} A^i D_i v_{\pm}, \tag{19}$$

$$R_{ij} = 2\Theta^{-2}(D_{(i} v_{-)}(D_{j)} v_{+}). \tag{20}$$

We now state (modified versions of) two lemmas of Meyers [17] which we require in the proof of Lemmas 5 and 6. Lemma 3 is a special case of Corollary 1 to Theorem 1 of [17], whereas Lemma 4 is a special case of Lemma 5 of [17], combined with the corollary mentioned above.

Lemma 3. *On an asymptotically flat end \mathcal{N}^{∞} , let $\psi \in C^2$ be a solution of*

$$g^{ij}(\partial_i \partial_j + a_i \partial_j)\psi = 0 \tag{21}$$

with $\psi = O(r^{-\epsilon})$ for some real ϵ and $a_i = O^1(r^{-1-\delta})$ for some $\delta > 0$. Then $\psi = O^2(r^{-\epsilon})$.

Lemma 4. *Let $p \in N \cup \{0\}$, $k \in N$, $0 < \epsilon < 1$ and $\lambda = O^k(r^{-2-p-\epsilon})$. Then, on a domain $R^3 \setminus B$, the equation $\Delta\phi = \lambda$ (with Δ denoting the flat Laplacian) has a solution $\phi_{\text{spec}} = O^{k+2}(r^{-p-\epsilon})$. Thus the general solution ϕ_{gen} which vanishes at infinity can be written as $\phi_{\text{gen}} = \phi_{\text{hom}} + \phi_{\text{spec}}$ where ϕ_{hom} is a solution of $\Delta\phi_{\text{hom}} = 0$ with terms of order r^{-q} , $\forall q \in N$ with $q \leq p$.*

We are now in the position to prove the main asymptotic result.

Lemma 5. *Let $(\mathcal{M}, g_{\mu\nu})$ satisfy conditions T1 and T4 of Theorem 1.*

Then on \mathcal{N}^{∞} there are coordinates \widehat{x}^i (with $\widehat{r}^2 = \delta_{ij}\widehat{x}^i\widehat{x}^j$) and constants m_{\pm} and m_i^{\pm} such that

$$v_{\pm} = \frac{m_{\pm}}{\widehat{r}} + \frac{m_i^{\pm}\widehat{x}^i}{\widehat{r}^3} + O^2\left(\frac{1}{\widehat{r}^3}\right), \tag{22}$$

$$\gamma_{ij} = \delta_{ij} + \frac{m_+m_-}{\widehat{r}^4}(\widehat{x}^i\widehat{x}^j - \widehat{r}^2\delta_{ij}) + O^2\left(\frac{1}{\widehat{r}^3}\right). \tag{23}$$

Proof. The first part of the proof which leads to (26) is a straightforward adaption of a result on the asymptotic behaviour of Killing vectors obtained in the 4-dimensional framework ([18], Proposition 2.2).

From (10), (12) and (16) we get

$$D_i D_j V = \partial_i \partial_j V - \Gamma_{ij}^k \partial_k V = O(r^{-2-\epsilon}). \tag{24}$$

We then define $h = V^2 + r^2 g^{ij} D_i V D_j V$ and the radial derivative, $d/dr = (x^i/r)\partial_i$. Using (24) and Schwarz’s inequality yields $|(d/dr)h| \leq 2Ch/r$ for some constant $C > 0$. Upon

integration, this gives $|h| \leq Dr^{2C}$ for some constant $D > 0$, and thus $\partial_i V = O(r^{C-1})$. Inserting the latter estimate in (24) gives $\partial_i \partial_j V = O(r^{C-2-\epsilon})$. Integrating again, we obtain

$$\partial_i V = E_i + O^1(r^{C-1-\epsilon}) \tag{25}$$

for some constants E_i . If $C \leq 1$ this can be written as $\partial_i V = E_i + O^1(r^{-\epsilon})$. The latter result can also be obtained if $C > 1$ (with some $\epsilon > 0$ possibly different from the ϵ used above) by iterating the integration of (24), i.e. by inserting (25) in (24) once again and integrating, and by repeating this procedure sufficiently often. One further integration then gives $V = E_i x^i + F + O^2(r^{1-\epsilon})$ where F is a constant. But since V is bounded by virtue of T4 and since ξ^μ has been normalized so that V tends to 1 at infinity, we have $E_i = 0$ and $F = 1$. This now allows us to improve the iterated integration of (24) till we end up with

$$V = 1 + O^2(r^{-\epsilon}), \tag{26}$$

where again $\epsilon > 0$ might differ from the ϵ 's used so far. We now pass to the metric $\gamma_{ij} = V^2 g_{ij}$ and use harmonic coordinates x^i with respect to γ_{ij} on \mathcal{N}^∞ . (Such coordinates exist and coincide with the coordinates x^i of the harmonic atlas (t, x^i) with respect to $g_{\mu\nu}$ introduced in Section 1.) Applying Lemma 3 to (14), (16) and (26), we conclude that

$$\omega = O^2(r^{-\epsilon}). \tag{27}$$

Apart from irrelevant signs and numerical factors, the remaining part of the proof is identical to the Lorentzian analysis [19] and will only be sketched. We write Eqs. (19) and (20) as

$$\Delta v_\pm = O(r^{-2-\epsilon}), \quad \Delta \gamma_{ij} = O(r^{-2-\epsilon}). \tag{28}$$

That is to say, we shift all deviations from the flat Laplacians to the right-hand side. Applying Lemma 4, we find that there exist constants m_\pm such that

$$v_\pm = \frac{m_\pm}{r} + O^2\left(\frac{1}{r^{1+\epsilon}}\right), \tag{29}$$

$$\gamma_{ij} = \delta_{ij} + O^2\left(\frac{1}{r^{1+\epsilon}}\right), \tag{30}$$

and we note that in (30) a homogeneous solution of the Laplace equation of order r^{-1} would be incompatible with the harmonic coordinate condition. The next step is to insert (29) and (30) into (19) and (20) and again to invert the Laplacians by a trivial explicit calculation and with the help of Lemma 4. In general there appears a homogeneous solution in γ_{ij} of order r^{-2} which can be removed by a suitable coordinate transformation (compatible with the harmonic coordinate condition). We thus arrive at (22) and (23) but with remaining terms of the form $O^2(r^{-2-\epsilon})$. The required falloff of $O^2(r^{-3})$ can be obtained by another straightforward iteration. \square

We remark that the iteration leading to (22) and (23) can in fact be continued to arbitrary high order as in the Lorentzian case [20].

We also note that in terms of the radial coordinate \widehat{r} employed here the Schwarzschild metric takes on the asymptotic form (1), whereas the \widehat{r} in the Taub-bolt metric in which the asymptotic form (22), (23) holds is related to the r used in (2) by $\widehat{r} = r - \frac{5}{4}|n|$.

We now introduce two scalars $\Omega_{(+)}$ and $\Omega_{(-)}$ and new metrics $g_{ij}^{(+)}$ and $g_{ij}^{(-)}$ on \mathcal{N}^∞ as follows (recall that $0 < \Theta < 1$ on \mathcal{N}^∞).

$$g_{ij}^{(\pm)} = \Omega_{(\pm)}^2 \gamma_{ij} = \left(\frac{1 \pm \sqrt{\Theta}}{2\sqrt{\Theta}} \right)^2 \gamma_{ij} = \frac{1}{16} \{ \sqrt{(1+V^2)^2 - \omega^2} \pm 2V \}^2 g_{ij}. \quad (31)$$

(Superscripts and subscripts $(+)$, $(-)$ and (\pm) on g_{ij} and Ω have nothing to do with the suffixes $+$, $-$ and \pm on v , m and m_i used before).

Lemma 6. *Let $(\mathcal{M}, g_{\mu\nu})$ satisfy conditions T1 and T4 of Theorem 1. Then $(\mathcal{N}, g_{ij}^{(+)})$ is asymptotically flat with vanishing mass. Assume further that the constants m_+ and m_- of Lemma 5 do not vanish. Then $(\mathcal{N}, g_{ij}^{(-)})$ has a compactification such that $\widetilde{\mathcal{N}} = \mathcal{N} \cup \Lambda$ where Λ is a point, and $g_{ij}^{(-)}$ has a C^2 -extension to Λ .*

Proof. Due to Lemma 5, the asymptotic behaviour of $\Omega_{(+)}$ and $\Omega_{(-)}$ is

$$\Omega_{(+)} = 1 + \frac{m^2 - n^2}{4\widehat{r}^2} + \mathcal{O}^2\left(\frac{1}{\widehat{r}^3}\right), \quad \Omega_{(-)} = \frac{m^2 - n^2}{4\widehat{r}^2} + \mathcal{O}^2\left(\frac{1}{\widehat{r}^3}\right), \quad (32)$$

where $m = \frac{1}{2}(m_+ + m_-)$ and $n = \frac{1}{2}(m_+ - m_-)$. The proof of the first part is trivial (see e.g. [14] for the definition of mass of an AF manifold). As to the second part, if $m_\pm \neq 0$ then in coordinates $\bar{x}^i = \widehat{r}^{-2} \widehat{x}^i$ it is easy to see (as in the Lorentzian case, [20]) that $g_{ij}^{(-)}$ has a C^2 -extension to the point “at infinity” Λ given by $\bar{x}^i = 0$. \square

We remark that, again as in the Lorentzian case [21,29], there is even an analytic compactification.

4. The theorem

Recall that ω_μ is curl-free (11) and hence there exists, locally on \mathcal{M} , a scalar ω defined by $\nabla_\mu \omega = \omega_\mu$. Using assumption T3 of the theorem, ω exists globally on \mathcal{M} (and hence also globally on \mathcal{N}) and is defined up to an additive constant. Since \mathcal{N} has a single end, we choose ω such that it vanishes at infinity. (It then coincides with the scalar ω defined only on \mathcal{N}^∞ in Section 3.)

We can now introduce the “Ernst potentials” $\mathcal{E}_+ = V^2 + \omega$ and $\mathcal{E}_- = V^2 - \omega$ which have the following global properties.

Lemma 7. *Let $(\mathcal{M}, g_{\mu\nu})$ satisfy conditions T1, T3 and T4 of Theorem 1. Then $-1 < \mathcal{E}_\pm \leq 1$. More specifically, either*

L1 *at least one of the potentials satisfies $\mathcal{E}_\pm = 1$ on \mathcal{N} or*

L2 *both potentials satisfy $-1 < \mathcal{E}_\pm < 1$ on \mathcal{N} .*

Proof. Ricci flatness implies the following equations for \mathcal{E}_\pm on \mathcal{N} and on \mathcal{M} , respectively,

$$D^2\mathcal{E}_\pm = V^{-2}D_i\mathcal{E}_\pm D^i\mathcal{E}_\pm \geq 0, \tag{33}$$

$$\nabla^2\mathcal{E}_\pm = V^{-2}\nabla_\mu\mathcal{E}_\pm\nabla^\mu\mathcal{E}_\pm \geq 0. \tag{34}$$

From the strong maximum principle (Theorem 3.5 in [22]) applied to (33) \mathcal{E}_\pm can only have a maximum inside \mathcal{N} if it is constant. Furthermore, since the fixed points of μ_τ are of course interior points of \mathcal{M} , the maximum principle applied to (34) in a neighbourhood of the fixed points excludes maxima at such points (on \mathcal{M} and hence also on $\widehat{\mathcal{N}}$). Since both potentials approach the value 1 at infinity, we have either $\mathcal{E}_\pm < 1$ or $\mathcal{E}_\pm = 1$ on \mathcal{N} . From this and from the identity $\mathcal{E}_\pm = 2V^2 - \mathcal{E}_\mp > -\mathcal{E}_\mp \geq -1$ the remaining statement of the lemma follows easily. \square

Note that by imposing T4 we require $\widehat{\mathcal{N}}$ to be the union of a compact set and an “end”. We do not yet know whether this property also holds for \mathcal{M} . Therefore, the maximum principle applied only to (34) globally on \mathcal{M} would not suffice to prove the lemma.

We are now in the position of proving Theorem 1.

Proof of Theorem 1. The proof has two parts, (1) and (2) which correspond to the two cases of Lemma 7.

(1) *L1 of Lemma 7 holds:* We can assume without loss of generality that $\mathcal{E}_- = 1$ on \mathcal{N} , which implies $v_- = 0$ and $v_+ = V^{-2} - 1$, and hence $A_i = 0$ and $\Theta = 1$. This means that Eqs. (19) and (20) become $D_i D^i V^{-2} = 0$ and $R_{ij} = 0$, i.e. γ_{ij} is locally flat. Using coordinates adapted to the Killing vector, the metric takes the local form

$$ds^2 = V^2(d\tau + \eta_i dx^i)^2 + V^{-2}\delta_{ij} dx^i dx^j. \tag{35}$$

The one-form $\eta = \eta_i dx^i$ on \mathcal{N} can be obtained from ω as any particular solution of $\text{curl}_\gamma \eta = -V^{-4}\text{grad}\omega$. Since $\mathcal{E}_- = 1$ we have $\omega = V^2 - 1$ and therefore $\text{curl}_\delta \eta = \text{grad} V^{-2}$. Let us now show that $(\mathcal{M}, g_{\mu\nu})$ must contain nuts, against our assumptions, or must be the four-dimensional Euclidean space.

Consider first the case when $\mathcal{L}_\mathcal{M}$ is empty. Then it follows from the definition of asymptotic flatness that $\mathcal{N} = \widehat{\mathcal{N}}$ is a complete Riemannian manifold. Since, moreover, γ_{ij} is flat on the asymptotically flat end \mathcal{N}^∞ , $(\mathcal{N}, \gamma_{ij})$ must be diffeomorphic to (R^3, δ_{ij}) . Furthermore, V^{-2} is well-defined everywhere on \mathcal{N} and solves the flat Laplace equation $\Delta V^{-2} = 0$. Hence V must be constant and equal to its asymptotic value 1. This implies $\eta_i dx^i = 0$ and therefore $(\mathcal{M}, g_{\mu\nu})$ is locally flat. But since \mathcal{M} is simply connected, the flatness of $(\mathcal{M}, g_{\mu\nu})$ follows.

We have thus shown that $(\mathcal{M}, g_{\mu\nu})$ must contain either nuts or bolts, or else the space is Euclidean. Assume that $(\mathcal{M}, g_{\mu\nu})$ contains a bolt \mathcal{B} with gravity κ . It is easy to see that $\nabla_\mu \xi_\nu \nabla^\mu \xi^\nu|_{\mathcal{B}} = 2\kappa^2$ and $\epsilon_{\mu\nu\alpha\beta} \nabla^\mu \xi^\nu \nabla^\alpha \xi^\beta|_{\mathcal{B}} = 0$. Using (12), we obtain that the Ricci tensor \mathcal{R} of the metric $g_{ij} = V^{-2}\delta_{ij}$ is

$$\mathcal{R} = \frac{3}{8V^4} \nabla_\mu \mathcal{E}_+ \nabla^\mu \mathcal{E}_+ = \frac{3}{8V^2} (2\nabla_\mu \xi_\nu \nabla^\mu \xi^\nu + \epsilon_{\mu\nu\alpha\beta} \nabla^\mu \xi^\nu \nabla^\alpha \xi^\beta), \tag{36}$$

where the first equality requires $\omega = V^2 - 1$ and the second equality is generally valid. Hence, \mathcal{R} must be singular on the bolt, which is impossible from Lemma 2. We can therefore conclude that L1 in Lemma 7 requires that the four-metric contains nuts or else $(\mathcal{M}, g_{\mu\nu})$ is flat.

(2) *L2 in lemma 7 holds:* We first observe, using Lemma 7, that the fields $v_{\pm} = (1 + \mathcal{E}_{\pm})^{-1}(1 - \mathcal{E}_{\pm})$ introduced on \mathcal{N}^{∞} in Section 3 are well defined and non-negative on all of \mathcal{N} , and the same applies to $\Theta = 1 - v_+v_- = 4V^2(1 + \mathcal{E}_{\pm})^{-1}(1 + \mathcal{E}_{\mp})^{-1}$.

We also note that the constants m_{\pm} in Lemma 5 cannot vanish. Assume, on the contrary, that e.g. $m_- = 0$. Assume also for the moment that m_i^- in (22) does not vanish. The term of order \hat{r}^{-2} in (22) containing this constant does not have a definite sign and dominates the expansion for large \hat{r} , which contradicts $v_- > 0$. Therefore, $m_i^- = 0$. Applying Lemma 4 to (19) we find that the leading term in the expansion of v_- must necessarily be a solution of the flat Laplace equation. Since all such solutions of order \hat{r}^{-p} , $p \geq 2$, change sign on \mathcal{N}^{∞} , the same argument as above leads to a contradiction unless $m_- \neq 0$, and in the same way we conclude that $m_+ \neq 0$. Hence Lemma 6 on the asymptotic structure applies.

We next show that $g_{ij}^{(\pm)}$ are regular on $\partial\mathcal{N} = \mathcal{L}_{\hat{\mathcal{N}}}$. Since $V\Omega_{(+)}|_{\partial\mathcal{N}} = V\Omega_{(-)}|_{\partial\mathcal{N}}$ the metrics ${}^2g_{ij}^{(\pm)} = g_{ij}^{(\pm)} - n_i^{(\pm)}n_j^{(\pm)}$ agree on the bolts $\partial\mathcal{N}$ (the unit normal vectors of $\partial\mathcal{N}$ with respect to $g_{ij}^{(\pm)}$ are denoted by $n_i^{(\pm)}$). We also notice that under a conformal rescaling $g'_{ij} = \Omega(V, \omega)^2 g_{ij}$, the extrinsic curvature k_{ij} of a two-dimensional submanifold S in (\mathcal{N}, g_{ij}) changes according to

$$k'_{ij} = \Omega k_{ij} + \Omega^{-2} ({}^2g'_{ij}) n^k \left(\frac{d\Omega}{dV} \mathcal{D}_k V + \frac{d\Omega}{d\omega} \mathcal{D}_k \omega \right), \tag{37}$$

where n^k is the unit outward normal of S with respect to g_{ij} .

Setting $S = \partial\mathcal{N}$ we have $k_{ij} = 0$ due to Lemma 2. We next insert $\Omega = V\Omega_{(\pm)}$ in (37) and note that (14) can be written as $\nabla^2\omega = 4V^{-1}\nabla_{\mu}V\nabla^{\mu}\omega$ on \mathcal{M} , and so $g^{ij}\mathcal{D}_iV\mathcal{D}_j\omega|_{\partial\mathcal{N}} = 0$. Hence the second fundamental forms k_{ij}^{\pm} of $\partial\mathcal{N}$ in $(\mathcal{N}, g_{ij}^{(\pm)})$ satisfy $k_{ij}^{(+)} = -k_{ij}^{(-)}$. Therefore we can glue together the two manifolds $(\mathcal{N}, g_{ij}^{(+)})$ and $(\tilde{\mathcal{N}}, g_{ij}^{(-)})$ along $\partial\mathcal{N}$ to obtain a C^2 manifold with C^1 metric [24]. Since the metric is piecewise C^2 , it follows that it is $C^{1,1}$. By Lemma 6, the resulting space is a complete three-dimensional asymptotically flat manifold with $C^{1,1}$ metric and vanishing mass. A short computation shows that it also has non-negative Ricci scalar, namely

$$R^{(\pm)}\Omega_{(\pm)}^4 = \frac{3 \pm 2\sqrt{\Theta}}{2\Theta^3} \gamma^{ij} A_i A_j \geq 0. \tag{38}$$

The rigidity part of the positive mass theorem [14] implies that this manifold must be diffeomorphic to R^3 with the flat metric. In particular, both metrics $g_{ij}^{(\pm)}$ are flat.

Expression (38) also shows that $A_i = 0$, which is equivalent to $v_+ = \alpha v_-$, where α is a positive constant (due to $v_{\pm} > 0$). It is convenient to introduce a function $H = (1 + \sqrt{\Theta})/(1 - \sqrt{\Theta})$ which is regular and satisfies $H|_{\mathcal{N}} > 1$ and $H|_{\partial\mathcal{N}} = 1$. Eq. (20) provides the Ricci tensor of the metric γ_{ij} in terms of the gradient of v_+ . Using (31) and

the fact that $g_{ij}^{(+)}$ is flat, the standard formula for the Ricci tensors of conformally related metrics gives (written in Euclidean coordinates):

$$\partial_i \partial_j H - \frac{\delta^{kl} (\partial_k H) (\partial_l H)}{2H} \delta_{ij} = 0, \tag{39}$$

Multiplying (39) with $\partial_j H$ we obtain $\delta^{ik} \partial_l H \partial_k H = \beta H$, where β is a constant. Since H cannot be constant it follows that $\beta > 0$, and we can write $\beta = 16M^{-2}$ for some positive constant M . Then (39) becomes $\partial_i \partial_j H = 8M^{-2} \delta_{ij}$ whose general solution can be written, after performing an appropriate translation $\tilde{x}^i = x^i + c^i$, as $H = 4M^{-2} \delta_{ij} \tilde{x}^i \tilde{x}^j$. The knowledge of H implies that of v_+ and $\Omega_{(+)}$. In spherical coordinates $\{\tilde{r}, \tilde{\theta}, \tilde{\phi}\}$, we have

$$v_+ = \sqrt{\alpha} \frac{4M\tilde{r}}{M^2 + 4\tilde{r}^2}, \quad ds_{\tilde{\gamma}}^2|_{\mathcal{N}} = \left(1 - \frac{M^2}{4\tilde{r}^2}\right)^2 (d\tilde{r}^2 + \tilde{r}^2 d\Omega^2), \quad \tilde{r} > M/2,$$

where we used $H|_{\partial\mathcal{N}} = 1$ and $H|_{\mathcal{N}} > 1$ to determine the range of \tilde{r} . We now define two constants $m > |n| \geq 0$ by $\alpha = (m+n)^{-1}(m-n)$ and $M = \sqrt{\alpha}(m+n)$ and perform the coordinate transformation $r = m + \tilde{r} + M^2/(4\tilde{r}^2)$. The metric γ_{ij} , V and ω take the form

$$ds_{\gamma} = dr^2 + (r^2 - 2mr + n^2) d\Omega^2, \quad r > m + \sqrt{m^2 - n^2}, \tag{40}$$

$$V^2 = \frac{r^2 - 2mr + n^2}{r^2 - n^2}, \quad \omega = \frac{2n(r - m)}{(r^2 - n^2)}. \tag{41}$$

Using this, the metric $g_{\mu\nu}$ can be reconstructed by solving $\text{curl}_{\gamma} \eta = -V^{-4} \text{grad } \omega$ as in part (1). Finally, global regularity of $g_{\mu\nu}$ on \mathcal{M} requires [4] that either $n = 0$ which gives the 1-parameter family of Schwarzschild instantons, or that $m = \frac{5}{4}|n|$ which gives the 1-parameter family of Taub-bolt instantons. \square

5. Discussion

We can think of improving our theorem in various directions. We discuss here briefly three of the most striking problems, namely:

- P1.** As mentioned already in Section 1, we believe that in T1 the assumption that \mathcal{N} is a manifold can be dropped.
- P2.** We would like to prove uniqueness of Taub-bolt purely in the four-dimensional setting, e.g. as formulated in Conjecture 1.
- P3.** It would be desirable to have a uniqueness result for the case in which nuts as well as bolts are a priori allowed to be present.

P1. We can show that \mathcal{N} , the space of non-trivial Killing orbits μ_{τ} of \mathcal{M} is in fact a manifold provided that all these orbits have the same period. In a neighbourhood of a fixed point p of μ_{τ} , the period of the orbits is determined by the “gravities” of the fixed point (cf. [1]). This is due to the commutativity of the isometry and the exponential map, viz. $\exp(\mu_{\tau*} X) = \mu_{\tau}(\exp X)$ for a tangent vector X at p , and to the fact that locally the exponential map is one-to-one. Since a bolt has only one “gravity”, all orbits in its

neighbourhood have the same period. More generally, on a Riemannian manifold with a bolt one has control over the period of the orbits on a domain in which the exponential map is non-singular. This domain is bounded by the cut points of the geodesics emanating from the bolt (cf. Section VIII, Theorem 7.4. of [25]). On these cut loci, the period might change and the manifold structure of \mathcal{N} may be lost, though we do not believe that this will happen.

P2. In the AF case, the conjecture can actually be proven rather straightforwardly along the lines of the Lorentzian case [7] (and leads to the Schwarzschild instanton (1)), by first showing an analog of the “staticity theorem” (cf. Section 8.2 of [7]), followed by the reasoning of the present paper restricted to the case of a hypersurface-orthogonal Killing vector.

As to the asymptotically locally flat case, we recall here from [6,28] the definition of ALF (with slight modifications, in particular using the cyclic group Z_Q instead of the more general options in [6,28]).

- (1) The “end” $\mathcal{M}^\infty \setminus \{\text{a suitably chosen compact set}\}$ is diffeomorphic to $R \times (S^3/Z_Q)$.
- (2) The lift $d\check{s}^2$ of the line element ds^2 to the covering space $R \times S^3$ takes the form

$$d\check{s}^2 = dr^2 + r^2(\sigma_1^2 + \sigma_2^2) + \sigma_3^2 + O^2(r^{-1}), \quad (42)$$

where $r \in (r_0, \infty)$ for some constant r_0 is a radial coordinate (in the R direction), σ_i ($i = 1, 2, 3$) are the left-invariant one-forms on the unit sphere.

If a 1-parameter group of isometries μ_τ of $(\mathcal{M}^\infty, g_{\mu\nu})$ is present, it is likely that the latter will leave the lens spaces S_r/Z_Q invariant (with S_r denoting the 3-spheres of constant radius $r \in (r_0, \infty)$), and that Z_Q will act on each of the S_r as a subgroup of the lift $\check{\mu}_\tau$ of μ_τ to the S_r .

If this is the case, we can show that the 1-form dual to the Killing field ξ^μ is parallel to σ_3 , and that the quotient manifold (\mathcal{N}, g_{ij}) is asymptotically flat, as defined in Section 3 and as required in our theorem.

We remark that the “end” of the Taub-bolt metric is $R \times S^3$. So we could simplify the asymptotic condition by requiring this topology instead of the lens space S_r/Z_Q .

P3. Theorem 1 only allows the presence of bolts. Assume we relax assumption **T2** by also allowing (or only allowing) the presence of nuts, but we keep the other requirements. Then there is the example of multi-Taub–NUT space (which has only nuts) and which might well be the only example. Since this space has a conformally flat space of orbits, it might then be possible to show its uniqueness by the strategy of Bunting and Masood-ul-Alam [13] as pursued in the present paper. If we allow bolts a priori, the problem with this strategy is to find a suitable conformal rescaling of the metric on \mathcal{N} which in particular keeps the metric regular on the bolts. In general a rescaling might shrink the bolts to points or shift them to infinity. There is nevertheless hope to get sufficient control also in this general case in order to apply a positive mass theorem.

On the other hand, the uniqueness result for the Euclidean Kerr metric [2] is restricted to the asymptotically flat case, allows only nuts and effectively restricts their number to 2 by requiring that $\mathcal{M} = R^2 \times S^2$. We note that in this case the space of orbits is not a manifold (contrary to the statement in [2]; the proof of [2] needs to be and can easily be rewritten accordingly). The strategy of this proof is based on a generalization of Israel’s proof of

uniqueness of the Schwarzschild solution, and on a suitable characterization of the Kerr metric. In fact, this characterization naturally extends to the asymptotically locally flat class of “Kerr–Taub-NUT” instantons [2] (called “Kerr–Taub-bolt” metrics by the discoverers [26,29] since they refer to another Killing vector). Assuming ALF instead of AF it should be possible to extend the uniqueness proof of [2] to this class. While we also believe that the topological conditions of the original proof can be relaxed, they can probably not be removed altogether.

It would also be interesting to prove uniqueness of asymptotically (locally) Euclidean instantons [6] along these lines.

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